Calculus: Convergence of Series

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Contents

0 Preface

The following lecture notes are mostly based on lecture videos provided by the lecturer. Any mistakes in the following notes are likely to be the author's. Further reading and practice problems are highly encouraged.

1 Convergent Infinite Series

Given an infinite sequence $(a_1, a_2, a_3, ...)$, the nth partial sum S_n is the sum of the first n terms of the sequence. That is,

$$S_n = \sum_{k=1}^{n} a_k$$

An infinite series is said to be **convergent** if the sequence of its partial sums $(S_1, S_2, S_3, ...)$ tends to a limit. Mathematically, a series converges if there exists a number ℓ such that for every arbitrarily small positive number ε, there is a (sufficiently large) integer N such that for all $n \geq N$,

$$|S_n - \ell| < \varepsilon$$

A series that is not convergent is said to be divergent. There are multiple tests to find out whether a series is convergent or divergent.

2 Ratio Test

Given a series

$$\sum_{n=1}^{\infty} a_n$$

where each term is a real or complex number and a_n is nonzero when n is large. The ratio test uses the limit

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

where a_n and a_{n+1} corresponds to the nth and $(n+1)$th term of the series.

There are three cases for the value of L:

- if $L < 1$ then the series converges absolutely.
- if $L > 1$ then the series is divergent.
- if $L = 1$ or the limit fails to exist, then the test is inconclusive, because there exist both convergent and divergent series that satisfy this case.

2.1 Example Problem

Given the infinite series

$$\sum_{n=1}^{\infty} \frac{2^n}{n!\,n}$$

Find out whether it is convergent.

Solution

We begin by finding the nth and the $(n+1)$th term of the series

$$a_n = \frac{2^n}{n!\,n}$$
$$a_{n+1} = \frac{2^{n+1}}{(n+1)!\,(n+1)}$$

Then, by the ratio test, find the absolute value of the ratio between the two terms

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{2^{n+1}}{(n+1)!\,(n+1)} \cdot \frac{n!\,n}{2^n} \right|$$

Using basic indices rule and the fact that

$$(n+1)! = (n+1)\,n!$$

We can simplify L into

$$L = \lim_{n\to\infty} \left| \frac{2\, n!\, n}{(n+1)\, n!\, (n+1)} \right|$$

$$= \lim_{n\to\infty} \left| \frac{2\, n}{(n+1)\, (n+1)} \right|$$

$$= \lim_{n\to\infty} \left| \frac{2\, n}{(n+1)^2} \right|$$

Direct substitution of the limit will result in an indeterminate form of $\frac{\infty}{\infty}$, thus the usage of **L'Hôpital's rule** is required.

$$L \overset{\text{L'H}}{=} \lim_{n\to\infty} \left| \frac{2}{2(n+1)} \right|$$

Finally, substituting the limit will result to a finite number

$$L = \frac{2}{\infty}$$
$$L = 0$$

Since $L < 1$, we can conclude that the series is convergent.

3 Limit Comparison Test

Given two series $\sum_n a_n$ and $\sum_n b_n$ with $a_n \geq 0, b_n > 0$ for all n. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

with $0 < L < \infty$, then either both series converge or both series diverge.

We can utilize this test by having prior knowledge of the convergence of another series. Later on, we can set this series as $\sum_n b_n$ and compare it to the series in question, $\sum_n a_n$.

We'll introduce two series which we can utilize.

3.1 Geometric Series

The general form of a geometric series is given by

$$\sum_{n=0}^{\infty} ar^n$$

where r corresponds to the ratio between two consecutive terms. We can tell whether the geometric series is convergent by observing its ratio. The two cases of r are:

- if $|r| < 1$ then the series converges.
- if $r > 1$ or $r < -1$ then the series is divergent.

3.2 p-Series

The p-series, or the hyperharmonic series, is defined as

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

for any real number p.

When $p = 1$, the p-series is the harmonic series and is divergent. Else, p-series converges for all $p > 1$.

Note: If $p > 1$ then the sum of the p-series is $\zeta(p)$, i.e., the Riemann zeta function evaluated at p.

3.3 Example Problem

Given the infinite series

$$\sum_{n=1}^{\infty} \frac{3n^2 + 1}{2n^5 + n + 2} \tag{1}$$

Find out whether it is convergent.

Solution

Firstly, we need to find a suitable series to set as $\sum_n b_n$. A good choice would be a series that is not only simple, but allows us to evaluate the limit easily. By setting

$$\sum_n b_n = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

We do know that the series converges since it is a p-series with $p = 3$. Since $p > 1$, the series therefore converges.

From there, we can carry out the limit comparison test with summation (1) as our $\sum_n a_n$.

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}$$

$$= \lim_{n\to\infty} \frac{3n^2 + 1}{2n^5 + n + 2} \cdot \frac{n^3}{1}$$

$$= \lim_{n\to\infty} \frac{3n^5 + n^3}{2n^5 + n + 2}$$

With the same degree of polynomial on both the numerator and denominator of the fraction, we can conclude that the limit L tends to

$$L = \frac{3}{2}$$

Since $L > 0$, then we can conclude that both series converges and that the infinite series (1) converges.

4 Integral Test

Consider an infinite series

$$\sum_{n=N}^{\infty} f(n)$$

where N is an integer and f is a non-negative function defined on the unbounded interval $[N, \infty)$, on which it is monotone decreasing. The infinite series converges to a real number if and only if the improper integral

$$\int_{N}^{\infty} f(x)dx$$

is finite. Else, the infinite series diverges just like its integral.

4.1 Example Problem 1

Given the infinite series

$$\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^{\frac{3}{2}}} \tag{2}$$

Find out whether it is convergent.

Solution

To conveniently carry out the integral test, we replace all n with x and set the terms in summation as $f(x)$.

$$f(x) = \frac{x}{(x^2+1)^{\frac{3}{2}}}$$

Then integrate the function with the corresponding bounds

$$I = \int_{1}^{\infty} f(x)\,dx$$

$$= \int_{1}^{\infty} \frac{x}{(x^2+1)^{\frac{3}{2}}}\,dx$$

By letting $u = x^2 + 1$, we find that

$$u = x^2 + 1$$
$$\frac{du}{dx} = 2x$$
$$\frac{1}{2}\,du = x\,dx$$

Also notice that the upper bound of integration do not change after substitution, whereas its lower bound becomes 2. Thus we obtain

$$
\begin{aligned}
I &= \frac{1}{2}\int_{2}^{\infty} \frac{1}{u^{\frac{3}{2}}}\,du \\
&= \frac{1}{2}\int_{2}^{\infty} u^{-\frac{3}{2}}\,du \\
&= -u^{-\frac{1}{2}}\Big]_{2}^{\infty} \\
&= -\frac{1}{\sqrt{u}}\Big|_{2}^{\infty} \\
&= \lim_{u\to\infty}\left[-\frac{1}{\sqrt{u}}\right] - \left[-\frac{1}{\sqrt{2}}\right] \\
&= \frac{1}{\sqrt{2}}
\end{aligned}
$$

Since the integral is finite, series (2) therefore converges.

4.2 Example Problem 2

Given the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \tag{3}$$

Find out whether it is convergent.

Solution

Set $f(x)$ as the terms in summation (3)

$$f(x) = \frac{1}{4x^2 - 1}$$

Then integrate the function with the corresponding bounds

$$
\begin{aligned}
I &= \int_{1}^{\infty} f(x)\,dx \\
&= \int_{1}^{\infty} \frac{1}{4x^2 - 1}\,dx
\end{aligned}
$$

Utilizing partial fractions, we can split the integral into the following:

$$I = \frac{1}{2}\int_1^\infty \frac{1}{2x-1} - \frac{1}{2x+1}\, dx$$

$$= \frac{1}{2}\int_1^\infty \frac{1}{2x-1}\, dx - \frac{1}{2}\int_1^\infty \frac{1}{2x+1}\, dx$$

$$= \left[\frac{1}{4}\, ln|2x-1| - \frac{1}{4}\, ln|2x+1|\right]_1^\infty$$

$$= \left[\frac{1}{4}\, ln\left|\frac{2x-1}{2x+1}\right|\right]_1^\infty$$

$$= \lim_{x\to\infty}\left[\frac{1}{4}\, ln\left|\frac{2x-1}{2x+1}\right|\right] - \frac{1}{4}\, ln\left(\frac{1}{3}\right)$$

The limit of the term inside the natural logarithm evaluates to 1 as x tends to ∞, thus we are left with

$$I = \frac{1}{4}\, ln(1) - \frac{1}{4}\, ln\left(\frac{1}{3}\right)$$

$$= -\frac{1}{4}\, ln\left(\frac{1}{3}\right)$$

$$= \frac{1}{4}\, ln(3)$$

Since I evaluates to a finite number, summation (3) therefore converges.