Calculus: Linear First Order Differential Equations

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Contents

0 Preface

The following lecture notes are mostly based on lecture videos provided by the lecturer. Any mistakes in the following notes are likely to be the author's. Further reading and practice problems are highly encouraged.

1 Linear First Order Differential Equations

Given an ordinary differential equation in a form,

$$\frac{dy}{dx} + p(x)y = g(x) \tag{1}$$

It is said to be a linear and first order ordinary differential equation (ODE). To solve linear ODEs of form (1), we require the introduction of an integrating factor.

1.1 Integrating Factor

Integrating factor is a function that is utilized when solving differential equations to transform it into a simpler form to solve. Usually it is denoted by μ and is a function of the dependent variable.

Without pre-given proof, the integrating factor to be used to solve linear ODEs of form (1) is

$$\mu(x) = e^{\int p(x)dx} \tag{2}$$

1.2 Linear ODE Solution

Still without pre-given proof, we claim that the solution to ODEs of form (1) is:

$$y = \frac{1}{\mu}\left[\int \mu \cdot g \, dx + c\right] \tag{3}$$

where c is the constant of integration. Once the integral on the right-hand side is solved, we arrive at the solution of the ODE.

2 Proofs for Integrating Factor and Solution

If the reader believes claims (2) and (3) and is not interested for their proofs, then they can simply skip this section.

First, assume that (2) is the integrating factor to be used. We would like to show that by introducing the integrating factor μ we will find an alternative form of (1) that will later arrive in solution (3). Given an equation of the form:

$$y = g(x)$$

Multiplying both sides of the equation by the integrating factor μ will result in:

$$\mu \cdot y = \mu \cdot g(x)$$

If we differentiate only the left hand side of the equation, namely,

$$\frac{d}{dx}(\mu \cdot y) = \mu \cdot g(x) \tag{4}$$

We will need to apply the product rule, thus obtaining

$$\frac{d\mu}{dx} \cdot y + \frac{dy}{dx} \cdot \mu = \mu \cdot g(x) \tag{5}$$

Notice that the derivative of μ in (2) with respect to the dependent variable will result in:

$$\frac{d\mu}{dx} = \frac{d}{dx}\left(e^{\int p(x)dx}\right)$$

$$\frac{d\mu}{dx} = \frac{d}{dx}\left(\int p(x)\, dx\right) \cdot e^{\int p(x)dx}$$

$$\frac{d\mu}{dx} = p(x) \cdot e^{\int p(x)dx}$$

$$\frac{d\mu}{dx} = p(x) \cdot \mu$$

Substituting our result back to (5):

$$p(x) \cdot \mu \cdot y + \frac{dy}{dx} \cdot \mu = \mu \cdot g(x)$$

Factoring μ and cancelling them out:

$$\mu \left[p(x) \cdot y + \frac{dy}{dx} \right] = \mu \left[g(x) \right]$$

$$p(x) \cdot y + \frac{dy}{dx} = g(x)$$

is precisely the linear ODE in (1). With that, we can conclude that equation (4) is the same as our ODE problem in (1) and we can finally begin our proof on claim (3).

To arrive at solution (3), we only need to manipulate equation (4), namely,

$$\frac{d}{dx}(\mu \cdot y) = \mu \cdot g(x)$$

$$\int \frac{d}{dx}(\mu \cdot y)\, dx = \int \mu \cdot g(x)\, dx$$

$$\mu \cdot y = \int \mu \cdot g(x)\, dx$$

$$y = \frac{1}{\mu} \left[\int \mu \cdot g(x)\, dx \right]$$

and arrive at solution (3).

Therefore, by using the integrating factor in (2), we can conclude that equation (3) is a solution to ODEs of form (1).

3 Example Problems

3.1 Problem 1

Find the particular solution to the differential equation

$$t\,\frac{dy}{dt} + 2y = t^2 - t + 1$$

given that $y(1) = \frac{1}{2}$.

Solution

Firstly, we need to make the differential equation given to the form in (1).

$$t\,\frac{dy}{dt} + 2y = t^2 - t + 1$$

$$\frac{dy}{dt} + \frac{2}{t}\,y = t - 1 + \frac{1}{t}$$

Once in the correct form, we need to find the integrating factor μ:

$$\mu = e^{\int \frac{2}{t} dt}$$

$$\mu = e^{2\, ln|t|}$$

$$\mu = t^2$$

Lastly, solve for the general solution y,

$$y = \frac{1}{t^2}\left[\int t^2\left(t - 1 + \frac{1}{t}\right)\, dt\right]$$

$$y = \frac{1}{t^2}\left[\int t^3 - t^2 + t\, dt\right]$$

$$y = \frac{1}{t^2}\left[\frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2 + c\right]$$

To obtain the particular solution, we need to find the value of the constant c with the given boundary conditions,

$$\frac{1}{2} = \frac{1}{1^2}\left[\frac{1}{4}(1^4) - \frac{1}{3}(1^3) + \frac{1}{2}(1^2) + c\right]$$

$$\frac{1}{2} = \frac{5}{12} + c$$

$$c = \frac{1}{12}$$

Therefore the particular solution to the differential equation is:

$$y = \frac{1}{t^2}\left[\frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2 + \frac{1}{12}\right]$$

3.2 Problem 2

Find the general solution to the differential equation

$$\frac{dy}{dx} = 2xy + xe^{x^2}$$

Solution

Since the differential equation is not in form (1), we need to manipulate it a little:

$$\frac{dy}{dx} = 2xy + xe^{x^2}$$
$$\frac{dy}{dx} - 2xy = xe^{x^2}$$

Once in that form, we can find μ:

$$\mu = e^{\int -2x dx}$$
$$\mu = e^{-x^2}$$

And solving for the general solution, y,

$$y = \frac{1}{e^{-x^2}} \left[\int e^{-x^2} (xe^{x^2}) \, dx \right]$$

$$y = e^{x^2} \left[\int x \, dx \right]$$

$$y = e^{x^2} \left(\frac{1}{2}x^2 + c \right)$$